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Coupled Map Lattice on Diffusion Limited Aggregate: Dynamics on a Random Fractal

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Abstract. While the coupled map lattice has been studied on a d-dimensional Euclidean lattice as well as on arbitrary complex networks, there are very few studies on fractal and they are devoted to deterministic fractals such as Sierpinski Gasket. We introduce coupled map lattices on a random fractal namely DLA[1]. There are two possible definitions depending on whether or not the sum of weights is constant. We study various maps such as logistic map, circle map, and tent map, in this context. For a DLA cluster, the number of neighbors of a given site may range from1-4. If the sum of weights of all coupled sites, i.e. neighbors and self, is not conserved, the bifurcation diagram depends on the number of neighbors of a given site. Thus the bifurcation diagram for a site with 2 neighbors is different from the bifurcation diagram for a site with 3 neighbors. We study coupled logistic maps, coupled circle maps and coupled tent maps in this context. For coupled logistic maps, we observe a band merging crisis in the case when the sum of weights is not conserved. We give a modified definition of local persistence to quantify this transition and observe continuously the changing persistence exponents for 2 and 3 neighbors.

INTRODUCTION

Dynamics of Coupled Map Lattices on Fractal Models

Coupled map lattices (CML) on Euclidian lattice in d-dimensions is a very well studied system. The studies are also conducted on mean-field models where the global coupling of the mean variable is assumed. Most investigated system is coupled logistic maps, followed by coupled circle maps. However, other maps are also studied. In 80's the fractals and fractal geometry of nature was in vogue. Surprisingly, there are very few investigations on dynamical systems on a fractal. The fractals have a form of connectivity which has interesting scaling with distance.
CMLs on such systems show a transition from spatial disorder to spatially uniform, temporal chaos when the coupling is varied. We note that generic behavior in simulations of the neural network, coupled oscillators, and coupled maps suggests of separation of nodes into regions (domains) of fixed point, oscillating and chaotic regions (domains). On a non-uniform substrate, activity in all parts of the underlying lattice may not be uniform.

The network connectivity can have a profound influence on the segregation of activity in networks. We note that such partially arrested can be obtained even in low-dimensional coupled map lattices. These are known as chimera states. In this work, we consider a fractal model which has been very well investigated by the surface physics community. Most of the dendritic branching is fractal [3]. We study a model known as diffusion-limited aggregate (DLA) which models this phenomenon. We note that even neurons are fractally connected. Dynamics on fractally coupled networks can occur in a wide range of physical systems. It has been proposed that the fractal coupling in neurons may be essential for the greater efficiency to perform higher-order computations.

Coupled map lattices, that are spatially homogeneous structures, are simple, computationally amenable dynamical networks that have a behavior similar to more complex models. It is now quite well known that a variety of interesting physical processes can take place on objects with fractal structures. The discrete diffusion coupling on fractal lattices is a natural form of coupling in CMLs. The dynamics of coupled maps on the Sierpinski gasket has been studied for the spectrum of eigenvalues, eigenvectors, stability, and bifurcation of synchronized states[4].

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Chaotic maps like logistic map display some essential features of neuronal dynamics, such as fixed-point, oscillatory, or chaotic behavior. These dynamics depend on the applied stimulus that is used to study clustering and coding in neural networks [5]. Chaotic temporal states of systems can exhibit long-range spatial order with temporal chaos; provided the systems are probabilistic, have long-range interactions and are statistically symmetric [6].
The synchronization has been studied from the perspective of eigenvalue of the Laplacian matrix on underlying connectivity. Wigner-May theorem helps us to understand stability of randomly coupled elements. Eigenvalues of fractal show an interesting hierarchical structure and the instabilities to spatially uniform state are different[6].

Diffusion-Limited Aggregation (DLA)

A model for metal–particle aggregation process, having the correlations measured, was studied by simulation by Witten and Sander. They found that the density correlations within the model aggregates fall off with distance with a fractional power law similar to the metal aggregates. The radius of gyration of the metal aggregates shows power-law behavior.

This fractal model is based on the Eden model where particles are added one at a time at random to sites adjacent to the occupied sites. The so formed compact cluster has density correlations independent of the distance in the limit of large size [7].But for the metal–particle aggregation process, Witten and Sander found that the metal aggregates had correlations that fell off as fractional power law of distance. The critical correlations here arise from the irreversible growth process. The DLA model can be said to be the discrete version of the Langer – Krumbhaar model of dendritic growth[8].

COUPLED MAPS ON DLA

The initial state is a seed particle at the origin of a lattice. A second particle is added at some random site at large distance from the origin. This particle walks randomly until it visits a site adjacent to the seed. Then the walking particle becomes part of the cluster. Another particle is now introduced at a random distant point, and it walks randomly until it joins the cluster, and so forth. If a particle touches the boundaries of the lattice in its random walk it is removed and another introduced [1]. Here, we simulate a DLA over 10^5 sites.
We define variable value $x_{i,j}(t)$ to the site (i,j) at time t and the evolution is given by.

$$x_{i,j}(t+1) = (1-\varepsilon)\mathrm{f}\left(x_{i,j}\,(t)\right) + \frac{\varepsilon}{4}\left(\mathrm{f}\left(x_{i+1,j}(\mathrm{t})\right) + \mathrm{f}\left(x_{i-1,j}(\mathrm{t})\right) + \mathrm{f}\left(x_{i,j+1}(\mathrm{t})\right) + \mathrm{f}\left(x_{i,j-1}(\mathrm{t})\right)\right) \qquad \textbf{(1)}$$

where the contribution from sites which are not on DLA cluster is taken as zero and they are not evolved in time.
Alternatively,

$$x_{i,j}(t+1) = \left(1 - N(i,j)\frac{\varepsilon}{4}\right)\,\mathrm{f}\left(x_{i,j}\,(t)\right) + \frac{\varepsilon}{4}\sum_{\eta(\mathrm{i,j})}\mathrm{f}\left(x_{\eta(i,j)}\,(t)\right) \qquad \textbf{(2)}$$

where N(i,j) is number of neighbors of site (i,j) on DLA cluster. Again the contribution from sites which are not on DLA cluster is taken as zero and they are not evolved in time.

To make the difference clear, consider a case in which site (i,j) has only two neighbors, (i+1,j) and (i,j-1) on the cluster. The evolution according to rule **(1)** will be

$$x_{i,j}(t+1) = (1-\varepsilon)\,\mathrm{f}\left(x_{i,j}\,(t)\right) + \frac{\varepsilon}{4}\left(\mathrm{f}\left(x_{i+1,j}(\mathrm{t})\right) + \mathrm{f}\left(x_{i,j-1}(\mathrm{t})\right)\right) \qquad \textbf{(3)}$$

while according to **(2)** will be

$$x_{i,j}(t+1) = \left(1-\frac{\varepsilon}{2}\right)\mathrm{f}\left(x_{i,j}\,(t)\right) + \frac{\varepsilon}{4}\left(\mathrm{f}\left(x_{i+1,j}(\mathrm{t})\right) + \mathrm{f}\left(x_{i,j-1}(\mathrm{t})\right)\right) \qquad \textbf{(4)}$$

It is clear that the sum of weights is not conserved in rule **(1)** while it is conserved in rule **(2)**. In rule **(1)** we certainly expect the evolution of a given site to depend on the number of neighbors while in rule **(2)** it may or may not be so. A typical DLA cluster generated using the above algorithm is shown in Fig. 1.We shall now have a look at the circle, logistic and tent maps on DLA clusters using dynamics in **(1)** and **(2)**.

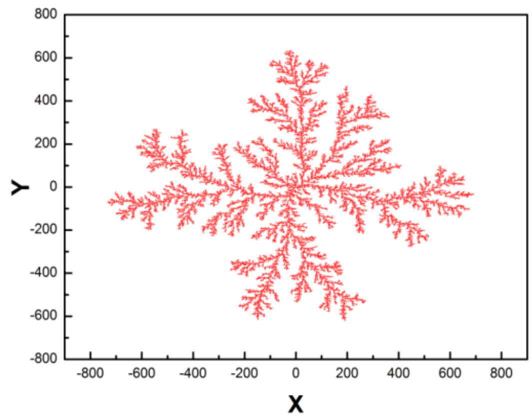


FIGURE. 1. A typical DLA cluster.

Circle Map, Logistic Map and Tent Map

The **circle map** is a one-dimensional map which maps a circle onto itself $\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\ \theta_n)$ where θ_{n+1} is computed mod 1. Here, there are two parameters Ω (externally applied frequency) and K (the strength of non-linearity). We consider K =1 and Ω = 0.065 . It is a prototypical map for studying phase dynamics [9]. The **logistic map** is defined as $x_{n+1} = \mu x_n(1 - x_n)$ where μ is a parameter and lies between 0 and 1. Study for a large value of n is desirable. For the logistic map, period-doubling bifurcations start for μ ≥ 3.3. For μ ≥ 3.56, we have a chaotic regime [9]. A piecewise linear, one-dimensional map known as **tent map** on the interval [0,1] exhibiting chaotic dynamics is given by $x_{n+1} = \text{r}\min(x_n, (1 - x_n))$.

We now plot the bifurcation diagrams, where we couple the sites on the DLA with the circle, logistic and tent maps for one neighbor, two neighbors,three and four neighbors. We define a control parameter **ε** that lies between 0 and 1.The bifurcation graphs are plots of coupled two-dimensional sites versus the control parameter.

First, we study the non-conserved case. Here, the sum of weights is not conserved and the bifurcation diagram depends on the number of the neighboring site/s.

Bifurcation Diagrams for Circle Map

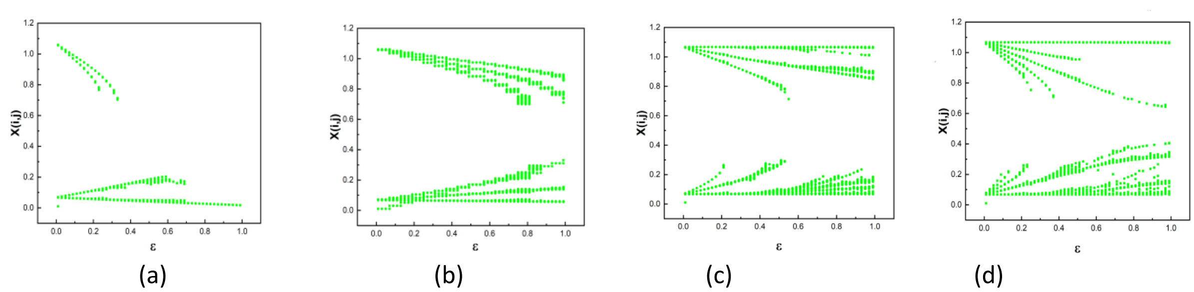
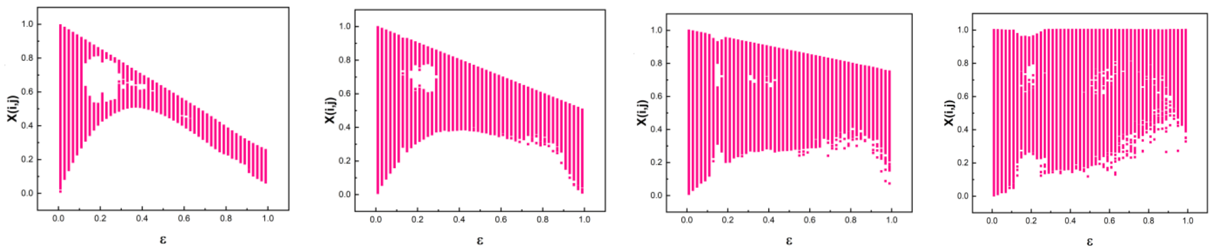


FIGURE. 2. The bifurcation diagram in the nonconserved case when the number of neighbors changes (a)sites with 1 neighbor (b)sites with 2 neighbors (c)sites with 3 neighbors (d)sites with 4 neighbors.

The sites on the DLA are coupled with a circle map. For one neighbor, the fixed point is stabilized for a larger value of coupling. In all cases, periodic behavior is obtained and there is no chaos for coupled circle maps.

Bifurcation Diagrams for Logistic Map

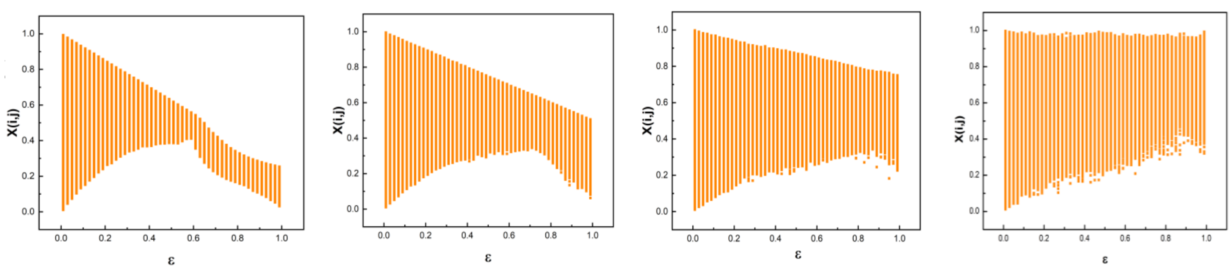


(a) (b) (c) (d)

FIGURE.3. Bifurcation diagram for nonconserved case (a) sites with 1 neighbor (b) sites with 2 neighbors (c) sites with 3 neighbors (d) sites with 4 neighbors. There are no periodic windows. But there is band-periodicity in a region.

The band attractor in the case of the logistic map appears different for the case of one neighbor; two, three and four neighbors concerned. It can be seen that the range of the band attractor with one neighbor is wider than the other three cases. The range of x values is smaller for fewer neighbors. This is a common feature in all coupled nonconserved cases.

Bifurcation Diagrams for Tent Map



(a) (b) (c) (d)

FIGURE. 4. . Bifurcation diagram for (a) sites with 1 neighbor (b) sites with 2 neighbors (c) sites with 3 neighbors (d) sites with 4 neighbors. There are no periodic windows or even band-periodicity is absent.

For tent maps, we observe no periodic windows or even coarse-grained periodicity. The coarse-grained periodicity will lead a non-compact bifurcation diagram. For any number of neighbors certain range of variable values is obtained without any gap. The range of values taken as well as maximum possible value increases as we increase the number of neighbors which is expected.

Bifurcation Diagrams for Circle, Logistic and Tent Map for Conserved Case

Now we see the conserved case. The sum of weights is conserved here and the number of neighboring site/s do not affect the bifurcation diagrams. They are in fact very similar for any number of neighbors for above-mentioned cases of the coupled circle, coupled logistic and coupled tent maps. So only one bifurcation diagram representing each map has been shown below.

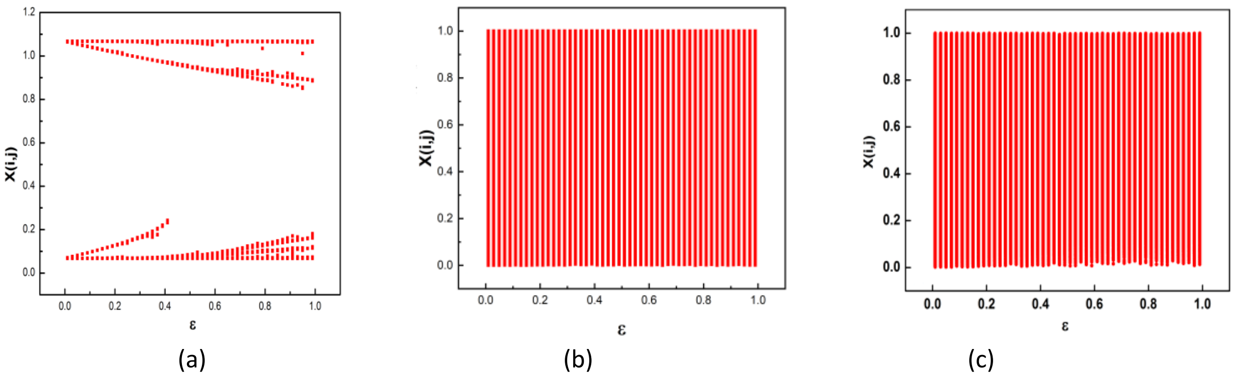


FIGURE. 5. Bifurcation diagram is almost independent of the number of neighbors for the conserved case. The bifurcation diagram is shown for (a) coupled circle maps (b) coupled logistic maps (c) coupled tent maps.

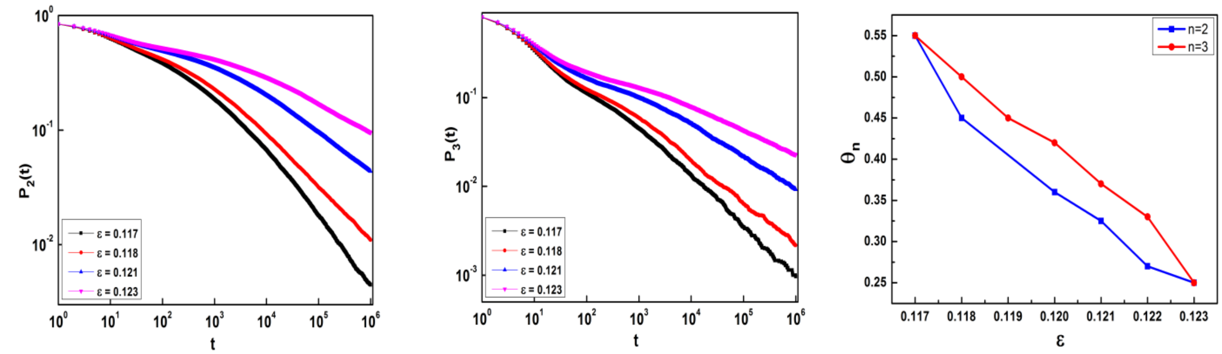
PERSISTENCE AS AN ORDER PARAMETER FOR THE FROZEN BAND-PERIODIC STATE IN COUPLED LOGISTIC MAPS

Persistence, a non- Markovian quantity, a concept which is a generalization of first passage time has been introduced in the context of statistical physics. Thus non-zero persistence implies that the system retains the memory of the initial conditions indefinitely. At the critical point, the persistence may have power-law decay and the decay exponent is called the persistence exponent. Persistence exponents are found to be a new class of exponents, which are not related to the usual critical exponents [10]. In the context of stochastic processes, it is defined as the probability P(t) that a stochastically fluctuating variable has not crossed a threshold value up to time t.

For spin systems, if the spin value does not change from its initial condition till time t, it is said to be persistent. The extension of this definition for map is usually carried out as follows. For a map, we map the variable values on spins by asserting that a site (i,j) has spin `up' if variable value $x_{i,j}(t) > x^*$ where $x^*$ is an unstable fixed point, and `down' spin otherwise. A site i is persistent if the associated spin value does not change until time t. We modify this definition slightly for the logistic map. We find the spin value at only even times. The reason is that the function has a negative slope at $x^*$and one expects the spin value to be back at even times. However, since the medium is inhomogeneous, the definition needs to be changed even further, particularly for a nonconserved case.

We define persistence as follows in a fractal. If a site (i,j) has only one neighbor, and if the site as well as its neighbor have same initial value, the value after first iteration will be $\left(1-\varepsilon+\left(\frac{\varepsilon}{4}\right)\right) f\left(x_{i,j}(0)\right)=\left(1-\frac{3\varepsilon}{4}\right)\mu\, x_{i,j}(0)\,(1-x_{i,j}(0))$. Thus it can be argued that effective value of parameter μ for a site with one neighbor is $\mu_1=\left(1-\frac{3\varepsilon}{4}\right)\mu$. Similarly $\mu_2=\left(1-\frac{\varepsilon}{2}\right)\mu$, $\mu_3=\left(1-\frac{\varepsilon}{4}\right)\mu$ and $\mu_4=\mu$. The effective unstable fixed points for these maps are $x_1^*=1-\frac{1}{\mu_1}$, $x_2^*=1-\frac{1}{\mu_2}$, $x_3^*=1-\frac{1}{\mu_3}$ and $x_4^*=1-\frac{1}{\mu_4}$. We compare the evolution with the unstable fixed point of the effective map, i.e. for a site with one neighbor we find if $x_{i,j}(t)> x_1^*$ and so on. Let $P_n^+(t)$ be a fraction of sites $x_{i,j}$ which had a value greater than the effective fixed point at t = 0 and $x_{i,j}(2t')> x_n^*$ for all t′≤t where n is the number of neighbors of the site. We define $P_n^-(t)$ as the fraction of sites $x_{i,j}$which had value less than the effective fixed point at t = 0 and $x_{i,j}(2t')< x_n^*$ for all t′≤t where n is the number of neighbors of the site. We define $P_n(t)=P_n^+(t)+P_n^-(t)$ .We find persistence for sites with different neighbors independently.

For all values of neighbors, the persistence goes to zero for smaller values of ε and saturates for larger values of ε. For very small values of ε, the persistence goes to zero exponentially fast. In all cases, there is an intermediate range of coupling for which we obtain slow decay of persistence. In particular, we obtain clear power-laws for n=2 and n=3 in the range 0.117≤ε≤0.123. We define neighbor-dependent persistence exponent θ_n as $P_n(t) \propto t^{-\theta n}$. The quantities $P_2(t)$ and $P_3(t)$ are plotted as a function of t for various values of ε in Fig.6 (a) and Fig.6 (b). Clearly the exponents θ_2 and θ_3 keep changing with ε. The variation of the exponent θ_n (for n=2 and n=3) with ε is plotted in Fig.6 (c).



(a) (b) (c)

FIGURE. 6. Plot of **(a)** $P_2(t)$ vs time(t) and **(b)** $P_3(t)$ vs time(t). **(c)** Plot of exponent θ_n vs control parameter ε.

CONCLUSIONS

We study dynamics on DLA which is a well-studied model of a random fractal. We study coupled circle map, logistic map and tent map on this system. We find a two-band attractor in a range of values of coupling in non-conserved case. We carefully define persistence and persistence exponent by taking into account the number of neighbors. We find that sites with 2 neighbors as well as sites with 3 neighbors show a continuously changing persistence exponent in a range of values of coupling parameter ε.

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