B. Sc. Part–II (Semester–IV) Examination

MATHEMATICS

(Modern Algebra : Groups and Rings)

Paper–VII

Time : Three Hours] [Maximum Marks : 60

Note : — (1) Question No. **1** is compulsory and attempt at once only.

(2) Solve **one** question from each unit.

1. Choose the correct alternatives : 1×10=10

(i) A non-empty subset H of the group G is a subgroup of G if and only if $a, b \in H \Rightarrow$

(a) $(ab)^{-1} \in H$ (b) $ab^{-1} \in H$

(c) $a^{-1}b^{-1} \in H$ (d) None of these

(ii) The product of an even and odd permutation is :

(a) Odd (b) Even

(c) Both odd and even (d) None of these

(iii) If G is a finite group and N is a normal subgroup of G, then O(G/N) is equal to :

(a) O(G).O(N) (b) O(G) + O(N)

(c) O(G)/O(N) (d) O(G) – O(N)

(iv) A group having only improper normal subgroup is called :

(a) A permutation group (b) A finite group

(c) A simple group (d) None of these

(v) If ϕ be a homomorphism of group G onto G' with kernel K, then G' is :

(a) Isomorphic to G/K (b) Isomorphic to K/G

(c) Isomorphic to G (d) One-one homomorphism

(vi) The Kernel of a homomorphism $f : G \rightarrow G'$ is :

(a) A normal subgroup of G (b) A subgroup of G'

(c) A normal subgroup of G' (d) None of these

(vii) A commutative ring without zero divisor is :

(a) Boolean Ring (b) Field

(c) Division Ring (d) None of these

(viii) If in a ring R, $x^2 = x \quad \forall x \in R$, then R is :

(a) Commutative ring (b) Division ring

(c) Boolean ring (d) Ring with unity

(ix) A ring R has maximal ideals :

(a) If R is finite (b) If R is finite with at least 2 elements

(c) Only if R is finite (d) None of these

(x) If R be commutative ring with unit element whose only ideals are $\{0\}$ and R itself. Then R is :

(a) A field (b) Division ring

(c) A proper ring (d) None of these

UNIT—I

2. (a) Prove that the system (G, t) is an abelian group with respect to 't' where $G = \{x / x = a + b\sqrt{2}, a, b \in Q\}$. 4

(b) If G is an abelian group, then prove that : $(ab)^n = a^n b^n \ \forall\ a, b \in G$ and $\forall$ integers n. 4

(c) Prove that the identity of a group G is unique. 2

3. (p) Let G be a group, then prove that $(ab)^{-1} = b^{-1} a^{-1} \ \forall\ a, b \in G$. 3

(q) Let H be a subgroup of a group G. For a, b $\in$ G, then prove that $Ha = Hb \Leftrightarrow a^{-1}H = b^{-1}H$. 3

(r) Prove that every subgroup of a cyclic group is cyclic. 4

UNIT—II

4. (a) Prove that the subgroup N of G is a normal subgroup of G if and only if the product of two right cosets of N in G is again a right coset of N in G. 4

(b) Let H be a subgroup of G. If $N(H) = \{g \in G / gHg^{-1} = H\}$, prove that N(H) is a subgroup of G. 4

(c) Show that if G is abelian then the quotient group G/N is also abelian. 2

5. (p) Let H be a subgroup of a group G. Let for $g \in G$, $gHg^{-1} = \{ghg^{-1} / h \in H\}$ prove that gHg^{-1} is a subgroup of G. 4

(q) If G is a group N is a normal subgroup of G, then show that G/N is also a group under the operation of multiplication of cosets. 4

(r) Show that every cyclic group is abelian. 2

UNIT—III

6. (a) If ϕ is a homomorphism of a group G into a group G' then prove that :

(i) $\phi(e) = e'$

(ii) $\phi(x^{-1}) = (\phi(x))^{-1}\ \forall x \in G$ 4

where e and e' are the unit elements of G and G' respectively.

(b) Let G be any group, g a fixed element in G . Define $\phi : G \to G$ by $\phi(x) = gxg^{-1}$. Prove that ϕ is an isomorphism of G onto G . 4

(c) Show that any kernel is nonempty. 2

7. (p) If ϕ is a homomorphism of G into G' with kernel K, then prove that K is a normal subgroup of G . 3

(q) Let G be the group of non-zero real numbers under addition and $G' = \{1, -1\}'$ where $1.1 = 1$, $1.(-1) = (-1)(1) = -1$, $(-1)(-1) = 1$. Define $\phi : G \to G'$ 3

Such that

$$\phi(x) = \begin{Bmatrix} 1, & x \text{ is positive} \\ -1, & x \text{ is negative} \end{Bmatrix}$$

Show that ϕ is a homomorphism.

(r) Show that the mapping $f : C \to R$ defined by $f(x+iy) = x$ is a homomorphism of the additive group of complex numbers onto the additive group of real numbers and find the kernel of f. 4

UNIT—IV

8. (a) If R is a ring in which $x^3 = x,\ \forall x \in R$, then prove that R is a commutative ring. 4

(b) Prove that the intersection of two subrings is a subring. 3

(c) Let the characteristics of the ring R be 2 and let ab = ba $\forall$ a, b $\in$ R. Then show that $(a+b)^2 = a^2 + b^2$. 3

9. (p) Define commutative ring. If R is a ring with zero element O, then for all a, b, c $\in$ R, prove that :

(i) a . O = O . a = O

(ii) a(−b)= (−a). b = −(ab)

(iii) (−a) (−b) = ab

(iv) a . (b − c) = a.b − a.c 1+4

(q) Let R be a ring with a unit element 1, in which $(ab)^2 = a^2b^2 \forall a, b \in R$. Prove that R must be commutative. 5

UNIT—V

10. (a) Prove that a homomorphism *f* of a ring R to a ring R' is an isomorphism iff $K_{erf} = \{0\}$. 4

(b) Let R be a commutative ring with unity. Prove that every maximal ideal of R is a prime ideal. 3

(c) If U is an ideal of a ring R with unity 1 and $1 \in U$, then prove that $U = R$. 3

11. (p) Let R and $\overline{R}$ be rings with zero elements O, $\overline{O}$ respectibvely and $f: R \to \overline{R}$ be a homomorphism. Then prove that : 5

(i) $f(O) = \overline{O}$

(ii) $f(-a) = -f(a) \forall a \in R$

(iii) $f(a-b) = f(a) - f(b) \forall a, b \in R$.

(q) If F is a field, then prove that its only ideals are $\{0\}$ and F itselt. 3

(r) Define : 2

(i) Left ideal

(ii) Simple Ring.