B.Sc. Part—III Semester—VI Examination

MATHEMATICS (Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Attempt **ONE** question from each Unit.

1. Choose the correct alternatives :

(1) If U and V are subspaces of a vector space W, then $U \cup W$ is a subspace of W iff :

(a) $U \supseteq W$ (b) $W \subseteq U$

(c) $U \subseteq W$ and $W \subseteq U$ (d) $U \subseteq W$ or $W \subseteq U$

(2) The basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of the vector space $R^3(R)$ is known as :

(a) Standard basis (b) Normal basis

(c) Quotient basis (d) None of these

(3) If $T : U \to V$ is the identity map, then Nullity of T is :

(a) 1 (b) 0

(c) 2 (d) 3

(4) The Kernel of a linear transformation $T : U \to V$ is a subset of :

(a) V (b) U

(c) U and V (d) None of these

(5) Annihilator of W, A(W) is a subspace of :

(a) W (b) V

(c) $\hat{V}$ (d) None of these

(6) Let U and V be complex vector spaces. If $A : U \to V$ be a linear map, then adjoint of A i.e. A* is a linear map :

(a) From $\hat{U}$ to $\hat{V}$ (b) From $\hat{V}$ to $\hat{U}$

(c) From U to V (d) From V to U

(7) Every set of orthogonal vectors is :

(a) Linearly Dependent

(b) Linearly Independent

(c) Linearly Independent and Linearly Dependent

(d) None of these

(8) If $\|U\| = 1$, then U is called :

(a) Normalised (b) Orthogonal

(c) Scalar inner product (d) Standard inner product

(9) The zero element of quotient module M/K is :

(a) M (b) K

(c) {0} (d) None of these

(10) Let M be an R-module, then M and {0} are called :

(a) Proper submodule of M (b) Zero module

(c) Improper submodule of M (d) None of these 1×10=10

UNIT—I

2. (a) Let R^+ be the set of all positive real numbers. Define the operations of addition $\oplus$ and scalar multiplication $\otimes$ defined on R^+ as follows :

(i) $u \oplus v = uv \quad \forall\, u, v \in R^+$

(ii) $\alpha \otimes u = u^\alpha \quad \forall\, u \in R^+, \alpha \in R$

Prove that R^+ is a real vector space. 5

(b) Define subspace of a vector space. Prove that intersection of two subspaces of vector space is a subspace. 5

3. (p) If S is a non-empty subset of a vector space V. Then prove that L(S) is the smallest subspace of V containing S. 5

(q) If U and W are two subspaces of a vector space V and Z = U + W, then show that $Z = U \oplus W \Rightarrow z = u + w$ uniquely for any $z \in Z$ and for some $u \in U$ and $w \in W$. 5

UNIT—II

4. (a) If T is a linear transformation from v_2 to v_2 defined by $T_{(2, 1)} = (3, 4)$, $T_{(-3, 4)} = (0, 5)$, then express (0, 1) as a L.C of (2, 1) and (–3, 4). Hence find image of (0, 1) under T. 5

(b) If $T : U \to V$ is a linear map, then prove that :

(i) N(T) is a subspace of U

(ii) R(T) is a subspace of V. 5

5. (p) State and prove Rank-Nullity theorem. 5

(q) Let $T : V_3 \to V_3$ defined by $T(x_1, x_e, x_3) = (x_1 + x_e, x_2 + x_3, x_3 - 2x_1)$. Find range, kernel, rank and nullity. Also verify Rank-Nullity theorem. 5

UNIT—III

6. (a) Let V be a vector space over F. For a subset S of V, let

$$A(S) = \{f \in \hat{V} : f(s) = 0, \forall s \in S\}$$

Prove that A(S) = A(L(S)), where L(S) is a linear span of S. 5

(b) If U,V are finite dimensional complex vector spaces and $A : U \to V$, $B : U \to V$ are linear maps $\alpha \in C$, then prove that :

(i) $(A + B)^* = A^* + B^*$

(ii) $(\alpha A)^* = \overline{\alpha} A^*$ 5

7. (p) Prove that eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent. 5

(q) If W is a subspace of finite dimensional vector space V, then prove that A(A(W)) = W. 5

UNIT—IV

8. (a) Define inner product space and if V is an inner product space, then prove that for arbitrary vectors $u, v \in V$ and scalar $\alpha \in F$:

(i) $||\alpha u|| = |\alpha| \, ||u||$

(ii) $||u + v|| \leq ||u|| + ||v||$ 5

(b) Let V be an inner product space over F. If $u, v \in V$, then prove that :

$$\langle u, v \rangle \leq ||u|| \cdot ||v||$$ 5

9. (p) Using Gram-Schmidt process orthonormalise the set of vectors

{(1, 0, 1, 0), (1, 1, 3, 0), (0, 2, 0, 1) in V_u. 5

(q) If $\{x_1, x_2, x_3, ... x_n\}$ is an orthogonal set, then prove that :

$$||x_1 + x_2 + x_3 + ... + x_n|| = ||x_1||^2 + ||x_2||^2 + ... + ||x_n||^2$$ 5

UNIT—V

10. (a) Define direct sum of submodules. If M_1 and M_2 are submodules of R-module M, then prove that $M_1 + M_2$ is a submodule of R-module M. 5

(b) Define homomorphism of modules and if $T : M \to H$ is an R-homomorphism, then prove that :

(i) $T_{(0)} = 0$

(ii) $T_{(-m)} = -m \quad \forall m \in M$

(iii) $T_{(m_1 - m_2)} = T_{(m_1)} - T_{(m_2)} \quad \forall m_1 m_2 \in M$ 5

11. (p) Define the submodule and prove that an arbitrary intersection of submodules of a module is submodule. 5

(q) If M is an R-module, H and K are submodules of M such that $K \subset H$, then prove that :

$$\frac{M}{H} = \frac{M/K}{H/K}$$ 5