B.Sc. Part—II (Semester—IV) Examination
MATHEMATICS
(Modern Algebra : Groups and Rings)
Paper—VII

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Solve **ONE** question from each unit.

1. Choose the correct alternatives (1 mark each) :

(1) Group G is abelian group if for all a, b $\in$ G :

(a) $a^{-1}b = ab^{-1}$ (b) $ab^{-1} = ab$

(c) $ab = ba$ (d) $ab = a^{-1}b$

(2) The set N of all natural numbers is a :

(a) Group (b) Subgroup

(c) Semi-group (d) Groupoid

(3) If N is a normal subgroup of a finite group G, then O(G/N) is equal to :

(a) $O(N)|O(G)$ (b) $O(G) \cdot O(N)$

(c) $O(G)|O(N)$ (d) None of these

(4) If N is a normal subgroup of G and H is a subgroup of G, then $H \cap N$ is :

(a) Subgroup (b) Cyclic group

(c) Normal subgroup (d) Simple group

(5) If $\phi : G \to G'$ is a homomorphism, then Ker ϕ is a :

(a) Subgroup of G′ (b) Subgroup of G

(c) Normal subgroup of G′ (d) Quotient group

(6) A homomorphism of a group into itself is :

(a) A homomorphism (b) An isomorphism

(c) An endomorphism (d) Monomorphism

(7) Let R be a ring with a unit element 1 and $(ab)^2 = a^2b^2 \ \forall \ a, b \in R$, then R is :

(a) Non-commutative ring (b) Division ring

(c) Simple ring (d) Commutative ring

(8) An integral domain is :

(a) Always a field (b) Never a field

(c) A field when it is finite (d) None of these

(9) If U is an ideal of a ring R with unity 1 and $1 \in U$, then :

(a) $U = M$ (b) $U \neq M$

(c) $U = R$ (d) $U \neq R$

(10) The intersection of two right ideals of R is :

(a) A right ideal of R (b) A left ideal of R

(c) Both left and right ideal of R (d) None of these 10

UNIT—I

2. (a) Define group and prove that cancellation laws hold in a group. 1+4

(b) Prove that the non-empty subset H of a group G, is a subgroup of G if and only if $a, b \in H \Rightarrow ab^{-1} \in H$. 5

3. (p) Show the intersection of two subgroups of a group is subgroup. Give an example to show the union of two subgroups of a group is not subgroup. 5

(q) Define permutation of a group. If $S = \{1, 2, 3, 4, 5\}$ and f, g are permutations on S given by :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

then prove that the product of f and g is not commutative. 1+4

UNIT—II

4. (a) If G is a group and N is a subgroup of index 2 in G, then prove that N is a normal group of G. 5

(b) If $G = \{1, -1, i, -i\}$ and $N = \{1, -1\}$, then show that N is a normal subgroup of multiplicative group G. Find quotient group G/N and also order of G/N. 5

5. (p) If N and M are normal subgroups of G, then prove that NM is also a normal subgroup of G. 5

(q) If G is a finite group and H is a subgroup of G, then prove that O(G)/O(N). 5

UNIT—III

6. (a) Define homomorphism. If $\phi : G \to G'$ is a homomorphism, then show that :

(i) $\phi(e) = e'$

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1} \ \forall x \in G$. 1+4

(b) If $\phi : G \to G'$ is a homomorphism with kernel K, then prove that $G/K \cong G'$. 5

7. (p) Show that if $\phi : G \to G'$ is a homomorphism with kernel K, then K is normal subgroup of G. 5

(q) If $\phi : G \to G'$ is a homomorphism with ker ϕ, then prove that ϕ is an isomorphism if and only if ker $\phi = \{e\}$; where 'e' is identity element of G. 5

UNIT—IV

8. (a) Prove that a non-empty subset K of a field F, is a subfield of F if and only if :

$$x - y,\ xy^{-1} \in K \quad \forall \ x, y \in K,\ y \neq 0.$$ 5

(b) Prove that an arbitrary intersection of subrings is a subring. 2

(c) Define :

(i) Ring with unity

(ii) Without zero divisor

(iii) Prime field. 3

9. (p) If R is a ring with unity and $(xy)^2 = x^2y^2 \ \forall \ x, y \in R$ then show that R is commutative. 5

(q) Prove that the characteristic of an integral domain is either zero or a prime number. 5

UNIT—V

10. (a) If U is an ideal of the ring R, then prove that R/U is a ring. 5

(b) If R is a commutative ring, with unit element whose only ideals are {0} and R, then prove that R is a field. 5

11. (p) Define :

(i) Maximal ideal

(ii) Principal ideal

(iii) Prime ideal.

Prove that intersection of two ideals is an ideal. 3+2

(q) If U is an ideal of a ring R, then prove that R/U is an homomorphic image of R. 5