B.Sc. Part—III Semester—VI Examination

MATHEMATICS

(Linear Algebra)

Paper—XI

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Attempt **ONE** question from each Unit.

1. Choose the correct alternatives :

(1) The basis {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of the vector space R(R) is known as :

(a) Normal basis (b) Quotient basis

(c) Standard basis (d) None of these

(2) Any super set of linearly dependent set is :

(a) Linearly dependent

(b) Linearly independent

(c) Linearly dependent or linearly independent

(d) None of these

(3) If $T : U \to V$ is an on to map, then :

(a) dim U < dim V (b) dim U > dim V

(c) dim U = dim V (d) dim U/dim V

(4) $\{T(u) | u \in U\}$ = ______.

(a) R(T) (b) N(T)

(c) R(u) (d) None of these

(5) An element of dual space of V is called a :

(a) Bilinear element (b) Linear functional

(c) Linear element (d) None of these

(6) If U and W are subspaces of V over F, then $U \subseteq W \Rightarrow$ ______.

(a) $A(U) = A(W)$ (b) $A(W) \not\subseteq A(U)$

(c) $A(U) \sim A(W)$ (d) $A(U) \supseteq A(W)$

(7) The normalized vector (1, –2, 5) is :

(a) (1, –2, 5)

(b) $\left(\frac{1}{2}, -1, \frac{5}{2}\right)$

(c) $\left(\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)$

(d) $\left(\frac{1}{5}, \frac{-2}{5}, 1\right)$

(8) In an inner product space V(F), following relation

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

is called :

(a) Schwartz's inequality

(b) Triangular law

(c) Parallelogram law

(d) Bessel's inequality

(9) If the ring R has a unit element 1 and $1.a = a, \ \forall \ a \in M$ is called :

(a) Right R-Module

(b) Left R-Module

(c) Unital R-Module

(d) None of these

(10) If $T : M \rightarrow H$ is a homomorphism of a R-module M into R-module H, then :

(a) R(T) is a submodule of M

(b) R(T) is a submodule of H

(c) N(T) is a submodule of H

(d) None of these 1×10=10

UNIT—I

2. (a) Show that a non-empty subset U of a vector space V(F) is a subspace of V iff :

(i) $u + v \in U \quad \forall \ u, v \in U$

(ii) $\alpha u \in U \quad \forall \ u \in U, \ \alpha \in F$ 5

(b) Define basis of a vector space. Prove that the set B = {(1, 1, 1), (1, –1, 1), (0, 1, 1)} is a basis of V_3. 1+4

3. (p) Prove that the intersection of two subspaces of a vector space is again a subspace. Is this statement true for union ? 3+2

(q) If U and W are two subspaces of a vector space V and Z = U + W, then show that $Z = U \oplus W \Leftrightarrow z = u + w$ uniquely for any $z \in Z$ and for some $u \in U$ and $w \in W$. 5

UNIT—II

4. (a) If $T : R^2 \rightarrow R^2$ is a linear map defined by T(3, 1) = (2, –4) and T(1, 1) = (0, 2), then express (7, 5) as a L.C. of (3, 1) and (1, 1). Hence find image of (7, 5) under T. 5

(b) State and prove Rank-Nullity theorem. 5

5. (p) Define linear transformation. If $T : U \to V$ is a linear map, then prove that :

(i) $T(0) = 0$

(ii) $T(-u) = -Tu \quad \forall\, u \in U$

(iii) $T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n) = \alpha_1 Tu_1 + \alpha_2 Tu_2 + \ldots\ldots + \alpha_n Tu_n$

$\forall\, \alpha_i \in F,\ u_i \in U,\ 1 \le i \le n$ and $n \in N$.

1+4

(q) Determine range, rank, kernel and nullity of linear map $T : V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_3 - 2x_1)$. 5

UNIT—III

6. (a) If V is finite dimensional vector space over F, then prove that $V \approx \hat{\hat{V}}$. 5

(b) If W_1 and W_2 are subspaces of a finite dimensional vector space V, describe $A(W_1 + W_2)$ in terms of $A(W_1)$ and $A(W_2)$. 5

7. (p) If W is a subspace of finite dimensional vector space V, then prove that :

$A(A(W)) = W$. 5

(q) Prove that Eigen vectors corresponding to distinct Eigen values of a square matrix are linearly independent. 5

UNIT—IV

8. (a) Define inner product space and if V is an inner product space, then prove that for arbitrary vectors $u, v \in V$,

(i) $\|u + v\|^2 + \|u - v\|^2 = 2\{\|u\|^2 + \|v\|^2\}$

(ii) $\|u + v\|^2 - \|u - v\|^2 = 4 \operatorname{Re}(u \cdot v)$. 1+2+2

(b) State and prove Cauchy-Schwartz inequality. 5

9. (p) Prove that $W^{\perp}$ is a subspace of V. 5

(q) Using Gram-Schmidt process orthonormalise the set of vectors

$\{(1, 1, 1), (0, 1, 1), (0, 0, 1)$ in V_3. 5

UNIT—V

10. (a) Define R-module homomorphism and if $T : M \to H$ is an R-module homomorphism, then prove that T is one-one $\Leftrightarrow K(T) = \{0\}$. 5

(b) Prove that every Abelian group G is a module over the ring of integer Z. 5

11\. (p) If R is a ring and $T : M \to H$ is an R-module homomorphism, then prove that :

$$\frac{M}{\text{Ker}(T)} \cong R(T)$$ 5

(q) Let M be an R-module then prove that :

(i) $r.0 = 0 \quad \forall\, r \in R$

(ii) $0.a = 0 \quad \forall\, a \in R$

(iii) $r(-a) = (-r)a = -ra$

(iv) $(-r)(-a) = ra$

(v) $r(a - b) = ra - rb \quad \forall\, r \in r, \; a, b \in M.$ 5